Math 252

Study Guide for Series Chapters

9.2 **Geometric Series**

Geometric series are characterized by terms that differ by a common ratio, *r*. Whereas *arithmetic* series are typified by terms that differ by a common difference.

Recall: $S_n = t_1\left(\frac{1-r^n}{1-r}\right) \rightarrow S = t_1\left(\frac{1}{1-r}\right)$ for $-1 < r < 1$

- See Chp 9.2: 9, 10; 24, 27, 36 – 39, 41, 47, 62
- *Geometric Series Problems* (gaskets) H/O
- *Chp 9 Sequences and Series* H/O
- *Series Problems* H/O

9.3 **Convergence of Series**

See Thm 9.2 for convergence properties – note that converses are rarely true.
Property 3 is the first convergence/divergence test listed on the Convergence H/O
This section introduces convergence tests like the Integral Test (Thm 9.3) and *p*-Test. It uses the Integral Test to show the divergence of the harmonic series, $\sum_{k=1}^{\infty}\frac{1}{k}$.

- See Chp 9.3: 1-9, 13-20, 25, 28, 33, 35, 37

9.4 **Tests for Convergence**

This section introduces the list of convergence tests summarized in the convergence H/O as well as the Alternating Series Error Bound (see 10.4).
For our class recall that for situations where it is appropriate, it will be sufficient to cite the Limit Comparison Test and list the comparison sequence (you will not need to show the limit of the ratio is finite).
Note Thm 9.9 Error Bounds for Alternating Series:

For $S_n = \sum_{i=1}^{n}(-1)^{i-1}a_i$, if $0 < a_{n+1} < a_n$ for all n and $\lim_{n\to\infty} a_n = 0$, then $|S - S_n| < a_{n+1}$

This says that if an alternating series converges, the difference between the convergent sum of the series and the sum of its first n terms is less than the next term in the sequence, a_{n+1}. That means the sum of the first n terms in the series is no more than a_{n+1} from the overall sum of the series.

- See Chp 9.4: 5-21 (any); 25-33; 37 -41, 57, 58, 62 – 89 (any), 107-109, 111

9.5 **Power Series and Interval of Convergence**

The power series centered at $x = a$: $\sum_{n=0}^{\infty} C_n(x-a)^n$ and, in particular, at $x = 0$: $\sum_{n=0}^{\infty} C_n x^n$ are introduced in this section. They are central to the idea of Taylor Series in chapter 10.
We use the Ratio Test to derive the interval of convergence since a power series will converge when $\lim_{n\to\infty}\left|\frac{C_{n+1}(x-a)^{n+1}}{C_n(x-a)^n}\right| < 1$ we have $|x-a|\lim_{n\to\infty}\left|\frac{C_{n+1}}{C_n}\right| < 1$ so the series converges when $|x-a| < \lim_{n\to\infty}\left|\frac{C_n}{C_{n+1}}\right|$ = R, where R is the radius of convergence.
The interval on convergence, therefore, is given by $a - R < x < a + R$. Note that the endpoints are not necessarily included. We test for those separately by substituting the values into the original series and testing for convergence using tests from 9.4.

- See Chp 9.5: 11-34 (any); 35-38; 40, 41-44, 46
- *Series Samples* H/O

10.1 **Taylor Polynomials**

Taylor Series are power series where the coefficients are based on the derivatives of the function being modeled. For the series centered at $x = a$:

$$S = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Similarly, for the series centered at $x = 0$ we have $S = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.

We recall that in previous work we used tangent line approximations to represent a function near a particular point. Taylor Series are a generalization of this approximation. A Taylor Polynomial of degree n is composed of the first n+1 terms of a Taylor Series:

$$P_n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Note that it is easy to confuse the polynomial center and the original function when $f(x)$ includes a power itself. The function $f(x) = \frac{1}{1+x}$ centered at $x = 0$, should have a Taylor Series that looks like $1 - x + x^2 - x^3 + x^4 - \cdots$ NOT $1 - (x+1) + (x+1)^2 - (x+!)^3 + (x+!)^4 - \cdots$

- See Chp 10.1: 1-13; 26 – 29; 46

10.2. **Taylor Series**

See note above regarding Taylor Series and recall radius of convergence from 9.5.

Four notable Taylor Series are observed in this section, three with an infinite radius of convergence:

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$

$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!} + \cdots$

$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^{n-1}\frac{x^{2n}}{(2n)!} + + \cdots$

Note that since these have an infinite radius of convergence, the center is inconsequential – so calculating it near $x = 0$ is most reasonable as it is simplest.

The one with a much smaller radius of convergence is the Binomial Series (also centered at $x = 0$):

$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots + \frac{p!}{(p-k)!k!}x^k + \cdots$ which has an interval of convergence of $-1 < x < 1$.

- See Chp 10.2: 1-7 (any); 10, 11, 18-22; 40-45; 47-51; 53
- *Taylor Series Problems* H/O

10.3 **Finding and Using Taylor Series**

This section introduces methods of generating Taylor Series that require only the elementary series (seen in 10.2) and substitution. As with all sections, please read the examples.

Typically we see that since $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$ it follows that

$$e^{-y^2} = 1 + (-y^2) + \frac{(-y^2)^2}{2!} + \frac{(-y^2)^3}{3!} + \cdots + \frac{(-y^2)^n}{n!} + \cdots = 1 - y^2 + \frac{y^4}{2!} - \frac{y^6}{3!} + \cdots + (-1)^{n-1}\frac{y^{2n}}{n!} + \cdots$$

- See Chp 10.3: 1-4; 16, 17, 30, 34, 36, 40, 41
- *Chp 10.1 Problems* H/O

10.4 **The Error in Taylor Polynomial Approximations**

Note the earlier observation (Theorem 9.9 in 9.4) regarding the Alternating Series Error Bound.
In general, the error bound for a Taylor polynomial of degree n on an interval $[a, b]$ is given by $E_n(x) = \frac{f^{(n)}(c)}{(n+1)!}(b)^n$ where the value of $f^{(n)}(c)$ is determined by the largest value of $f^{(n)}(x)$ attained on the interval $[a, b]$ (where $a \leq c \leq b$). In most cases, if $f^{(n)}(x)$ is an increasing function we choose $f^{(n)}(b)$ and if it is a decreasing function we choose $f^{(n)}(a)$. In the case where $f(x) = \sin(x)$ or similarly bounded functions, we often use the largest value of the function ($f(x) = 1$).
Note that for alternating series, such as that for $f(x) = \sin(x)$, the Alternating Series Error Bound is a reasonable, and often equivalent, error bound.

In many cases we are interested in finding the number of terms of the polynomial necessary to be within a particular margin of error. To find this value we solve $E_n(x) = \frac{f^{(n)}(c)}{(n+1)!}(b)^n$ where we have substituted a small error value for *E*.
See Example 3 for specific case.

- See Chp 10.4: 1-11, 13, 18-21